APPENDIX E
Special Functions

In this section, we collect some of the well-known properties of many of the special functions considered in this text. Most are quoted here without detailed proofs or derivations, while some have been discussed in a more physical context throughout the book.

E.1 Trigonometric and Exponential Functions

Although they are presumably familiar to all students, we briefly discuss the properties of the trigonometric and exponential functions. Since many of the special functions found in mathematical physics arise as solutions to similar differential equations, it is useful to recall here that:

- The differential equation

$$\frac{d^2f(x)}{dx^2} = -k^2 f(x) \tag{E.1}$$

  has the (conventionally normalized) trig function solutions $f(x) = \sin(kx), \cos(kx)$, while

- The differential equation

$$\frac{d^2f(x)}{dx^2} = +\kappa^2 f(x) \tag{E.2}$$

  has exponential solutions $f(x) = e^{\kappa x}, e^{-\kappa x}$.

The intuitive physical connections of these solutions with the oscillatory motion of a particle near a potential energy minimum (case (E.1)) versus the "runaway" (or damped) motion of a particle moved away from an unstable potential maximum (case (E.2)) can often be generalized to other differential equations to help understand the physical origin of the behavior of the solutions.

In each case above, as with any second-order ordinary differential equation, we obtain two, linearly independent solutions, $f_1(x)$, $f_2(x)$. The most general solution is then obtained by taking a linear combination $a_1 f_1(x) + a_2 f_2(x)$ and using the boundary conditions (in quantum mechanics) or initial conditions (in classical mechanics) to determine the arbitrary coefficients.

E.2 **Airy Functions**

The Airy differential equation is written in the form

$$\frac{d^2 f(x)}{dx^2} = xf(x) \tag{E.3}$$

Here we note the following:

- This problem is related to the quantum version of a particle moving under the influence of a uniform force.
- It also appears in the context of matching WKB-type (Chapter 10) solutions near classical turning points, where the potential energy function can be approximated (locally) as a linear potential.
- Using Eqn. (E.2) as a model, for $x > 0$, we expect exponentially damped or growing solutions; these should be consistent with the tunneling wavefunctions of Section 8.2.2.
- For $x < 0$ we expect oscillatory solutions with the period of oscillation *decreasing* for increasing $|x|$ as the "effective wave number" grows like $k^2 \sim |y|$.

The two linearly independent solutions are labeled $Ai(x)$ and $Bi(x)$ and are shown in Fig. E1. If we introduce the natural variable, $\zeta = 2x^{3/2}/3$, these solutions can be expanded for large values of $|x|$ as follows:

$$Ai(x) \longrightarrow \frac{1}{2}\frac{1}{\sqrt{\pi\sqrt{x}}} e^{-\zeta}\left[1 - \frac{c_1}{\zeta} + \cdots\right] \tag{E.4}$$

$$Ai(-x) \longrightarrow \frac{1}{\sqrt{\pi\sqrt{x}}}\left[\sin\left(\zeta + \frac{\pi}{4}\right) - \cos\left(\zeta + \frac{\pi}{4}\right)\frac{c_1}{\zeta} + \cdots\right] \tag{E.5}$$

$$Bi(x) \longrightarrow \frac{1}{\sqrt{\pi\sqrt{x}}} e^{\zeta}\left[1 + \frac{c_1}{\zeta} + \cdots\right] \tag{E.6}$$

$$Bi(-x) \longrightarrow \frac{1}{\sqrt{\pi\sqrt{x}}}\left[\cos\left(\zeta + \frac{\pi}{4}\right) + \sin\left(\zeta + \frac{\pi}{4}\right)\frac{c_1}{\zeta} + \cdots\right] \tag{E.7}$$

where $c_1 = 5/72$.

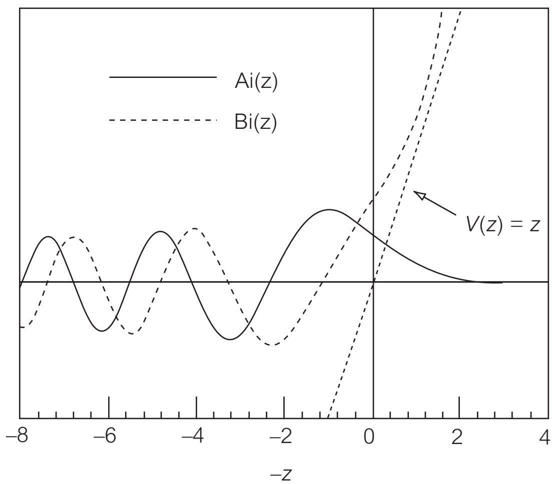

Figure E.1. Linearly independent solutions, $Ai(z)$ and $Bi(z)$, of the Airy differential equation.

E.3 **Hermite Polynomials**

The differential equation

$$\frac{d^2 h_n(z)}{dz^2} - 2z\frac{dh_n(z)}{dz} + 2nh_n(z) = 0 \tag{E.8}$$

is called *Hermite's equation* and has the solutions given by *Rodriges' formula*

$$h_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right) \tag{E.9}$$

which are polynomials of degree n. The solutions are defined over the interval $(-\infty, +\infty)$ and satisfy the normalization condition

$$\int_{-\infty}^{+\infty} [h_n(z)]^2\ e^{-z^2}\ dz = 2^n n!\sqrt{\pi} \tag{E.10}$$

The first few Hermite polynomials are given by

$$h_0(z) = 1 \qquad h_1(z) = 2z \qquad h_2(z) = 4z^2 - 2 \qquad h_3(z) = 8z^3 - 12z \tag{E.11}$$

E.4 **Cylindrical Bessel Functions**

The free-particle Schrödinger and wave equation in two dimensions (three dimensions), when written in polar (cylindrical) coordinates, leads to the equation

$$\frac{d^2 R_m(z)}{dz^2} + \frac{1}{z}\frac{dR_m(z)}{dz} + \left(1 - \frac{m^2}{z^2}\right) R_m(z) = 0 \tag{E.12}$$

where we consider integral values of m. The solutions are generically called *cylindrical Bessel functions*, and for each $|m|$, the two linearly independent solutions are labeled $J_{|m|}(z)$ (Bessel functions of the first kind) or $Y_{|m|}(z)$ (Neumann functions or Bessel functions of the second kind). Their limiting behavior and properties are discussed and displayed graphically in Section 15.3.1.

E.5 **Spherical Bessel Functions**

The free-particle Schrödinger equation in three-dimensions written in spherical coordinates yields another version of Bessel's equation, namely

$$\frac{d^2 R_l(z)}{dz^2} + \frac{2}{z}\frac{dR_l(z)}{dz} + \left(1 - \frac{l(l+1)}{z^2}\right) R_l(z) = 0 \tag{E.13}$$

with l an integer. Its solutions are the *spherical Bessel functions*, $j_l(z)$ and $n_l(z)$, which can be written in a standard form, in terms of the cylindrical Bessel functions

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z) \quad \text{and} \quad n_l(z) = \sqrt{\frac{\pi}{2z}} Y_{l+1/2}(z) \tag{E.14}$$

and are discussed in Section 16.6.

E.6 **Legendre Polynomials**

The (associated) Legendre's differential equation is written in the form

$$(1 - z^2)\frac{d^2\Theta_{l,m}(z)}{dz^2} - 2z\frac{d\Theta_{l,m}(z)}{dz} + \left(l(l+1) - \frac{m^2}{(1 - z^2)}\right)\Theta_{l,m}(z) = 0 \tag{E.15}$$

The solutions are the *associated Legendre polynomials* given by

$$P_l^m(z) = (-1)^m \frac{(1-z^2)^{m/2}}{2^l\, l!} \left(\frac{d}{dz}\right)^{l+m} \left(z^2-1\right)^l \tag{E.16}$$

for $m > 0$ and extended to negative m via

$$P_l^{-m}(z) = (-1)^m \frac{(l-m)!}{(l+m)!}\, P_l^m(z) \tag{E.17}$$

They are defined over the interval $(-1, 1)$ and the normalization is such that

$$\begin{aligned}\int_{-1}^{+1} dz\; P_l^m(z)\; P_{l'}^m(z) &= \int_0^{\pi} \sin(\theta)\, d\theta\; P_l^m(\cos(\theta))\; P_{l'}^m(\cos(\theta)) \\ &= \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}\, \delta_{l,l'} \end{aligned}\tag{E.18}$$

The special case of $m = 0$ gives the *Legendre polynomials* which are defined via

$$P_l(z) \equiv P_l^{m=0}(z) \tag{E.19}$$

which satisfy the differential equation

$$(1-z^2)\frac{d^2 P_l(z)}{dz^2} - 2z\frac{dP_l(z)}{dz} + l(l+1)\, P_l(z) = 0 \tag{E.20}$$

E.7 Generalized Laguerre Polynomials

The differential equation

$$\frac{d^2 G(z)}{dz^2} + \left(\frac{\alpha - 1}{z} + 1\right)\frac{dG(z)}{dz} + nG(z) = 0 \tag{E.21}$$

is called *Laguerre's equation* and has polynomial solutions labeled as

$$G(z) = L_n^{(\alpha)}(z) \tag{E.22}$$

which can be generated using *Rodrigues' formula*

$$L_n^{(\alpha)}(z) = \frac{e^z}{n!\, z^\alpha}\left(\frac{d}{dz}\right)^n \left[z^{n+\alpha}\, e^{-z}\right] \tag{E.23}$$

The solutions are defined over the interval $(0, +\infty)$ and satisfy the normalization condition

$$\int_0^{+\infty} dz\; z^\alpha\, e^{-z} \left[L_n^{(\alpha)}(z)\right]^2 = \frac{\Gamma(n+\alpha+1)}{n!} \tag{E.24}$$

E.8 The Dirac δ-Function

The Dirac δ-function was introduced and discussed extensively in Section 2.4 and here we only list some additional properties and identities. We recall that

$$\int_a^b dx\, f(x)\, \delta(x-c) = \begin{cases} f(c) & \text{if } a < c < b \\ 0 & \text{otherwise} \end{cases} \tag{E.25}$$

that is, the value of $f(x=c)$ is picked out from the integrand, or not, depending on whether the singularity is contained in the region of integration, or not. One can also derive (or justify) the following results.

$$\delta(ax) = \frac{1}{|a|}\, \delta(x) \tag{E.26}$$

$$\begin{aligned} \delta(x^2 - a^2) &= \delta[(x-a)(x+a)] \\ &= \frac{1}{|x+a|}\delta(x-a) + \frac{1}{|x-a|}\delta(x+a) \\ \delta(x^2 - a^2) &= \frac{1}{2|a|}\,(\delta(x-a) + \delta(x+a)) \end{aligned} \tag{E.27}$$

which is a special case of the more general relation

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|df/dx|_{x=x_i}} \tag{E.28}$$

where the sum is over all possible roots of $f(x_i) = 0$. Finally, recall that the *step-* or *Heaviside-function* is defined via

$$\Theta(x-a) = \begin{cases} 0 & \text{for } x < a \\ 1 & \text{for } x > a \end{cases} \tag{E.29}$$

and is given by

$$\Theta'(x-a) = \delta(x-a) \tag{E.30}$$

One can show that $\delta(x)$ can be obtained by taking the limit of the family of functions

$$\delta_\epsilon(x) = \frac{\epsilon}{\pi}\frac{\sin^2(x/\epsilon)}{x^2} \quad \text{as } \epsilon \to 0 \tag{E.31}$$

E.9 The Euler Gamma Function

Using integration by parts techniques, it is easy to derive Eqn. (D.34), namely

$$\int_0^\infty dx\, x^n\, e^{-x} = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \equiv n! \tag{E.32}$$

where $n!$ is read as "n-factorial." The integral can be generalized to noninteger values of n, and the *Gamma function* is defined in this way via

$$\int_0^\infty dx\, x^{n-1}\, e^{-x} \equiv \Gamma(n) \quad \text{for } n \neq 0, -1, -2, -3, \ldots \tag{E.33}$$

For positive integers, it reduces to the factorial function

$$\Gamma(n) = (n-1)! \quad \text{for integral } n > 0 \tag{E.34}$$

and also satisfies

$$\Gamma(n+1) = n\Gamma(n) \tag{E.35}$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)} \tag{E.36}$$

Other special values are

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2}\, dt = \sqrt{\pi} \tag{E.37}$$

which can be evaluated since this is now a Gaussian integral. This can be combined with Eqn. (E.36) (for nonnegative integral n) to give

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n} = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \tag{E.38}$$

which implicitly defines the double-factorial function. Finally, we note that *Stirling's formula* can be used to estimate the value of the factorial function for large argument, namely

$$\Gamma(n+1) = n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots\right) \tag{E.39}$$

E.10 Problems

PE.1. Show that the solutions to the Airy differential equation can be written in terms of cylindrical Bessel functions (satisfying Eqn. (E.12) of fractional ($n = \pm 1/3$)

order. For example, a standard result is that

$$Ai(-x) = \frac{1}{3}\sqrt{x}\left[J_{1/3}(y) + J_{-1/3}(y)\right] \tag{E.40}$$

where $y = 2x^{3/2}/3$.

PE.2. Estimate the value of 20! using Stirling's formula and compare to the exact value.